2.1 Tangent and Velocity Problems

Learning Objectives: After completing this section, we should be able to

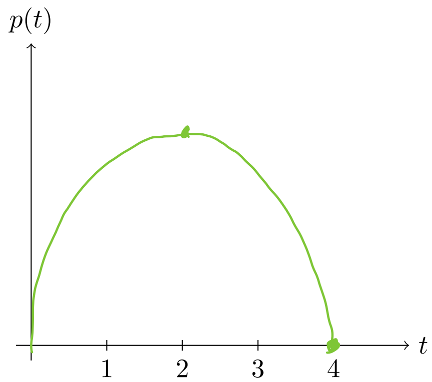
- approximate the slope of the tangent line to a curve at a point.
- approximate the instantaneous velocity of a moving object at a particular moment.

Driving question to start: If we know the exact position of on object, how can we find its velocity?

2.1.1 Limits

Example. Suppose we throw a baseball into the air. The function $p(t) = 64t - 16t^2$ gives the ball's height in feet at any time t seconds after throwing it. What is the velocity at $t = 1$ seconds?

Let's start with a graph:



When does the ball hit the ground?

$$p(t) = 0 = 64t - 16t^2$$
$$= 16t(4-t)$$

At $t = 0$ & 4 s, the ball is on the ground.

When does the ball reach its max height?

$t = 2$ seconds, by inspection.

Max height?

$$p(2) = 64 \cdot 2 - 16 \cdot 2^2 = 64 \text{ ft}$$

Can we first approximate the velocity? Let's find the **average velocity** over some time intervals.

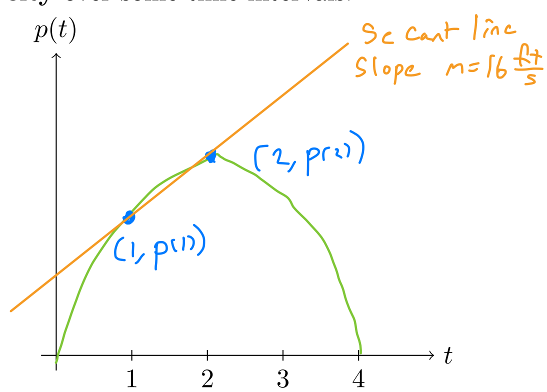
Average velocity between $t = 1$ s & $t = 2$ s is the slope of the secant line through $(1, p(1))$ & $(2, p(2))$

$$V_{avg} = \frac{\Delta p}{\Delta t} \quad \frac{\text{"change in position"}}{\text{"change in time"}}$$

"average velocity"

$$= \frac{p(2)-p(1)}{2-1} \frac{(\text{ft})}{(\text{sec})} = \frac{64 - (64(1) - 16(1)^2)}{2-1} \frac{\text{ft}}{\text{s}}$$

$$= 16 \frac{\text{ft}}{\text{s}}$$ = slope of secant line through $(1, p(1))$ and $(2, p(2))$

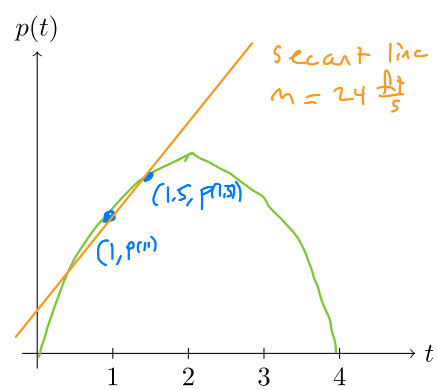


Can we get a better estimate for the velocity at $t = 1$ s? => yes, use a smaller interval

average velocity between $t = 1$ s and $t = 1.5$ s:

$$V_{avg} = \frac{\Delta p}{\Delta t} = \frac{p(1.5) - p(1)}{1.5 - 1} \frac{\text{ft}}{\text{s}}$$

$$= \frac{60 - 48}{0.5} \frac{\text{ft}}{\text{s}} = 24 \frac{\text{ft}}{\text{s}}$$



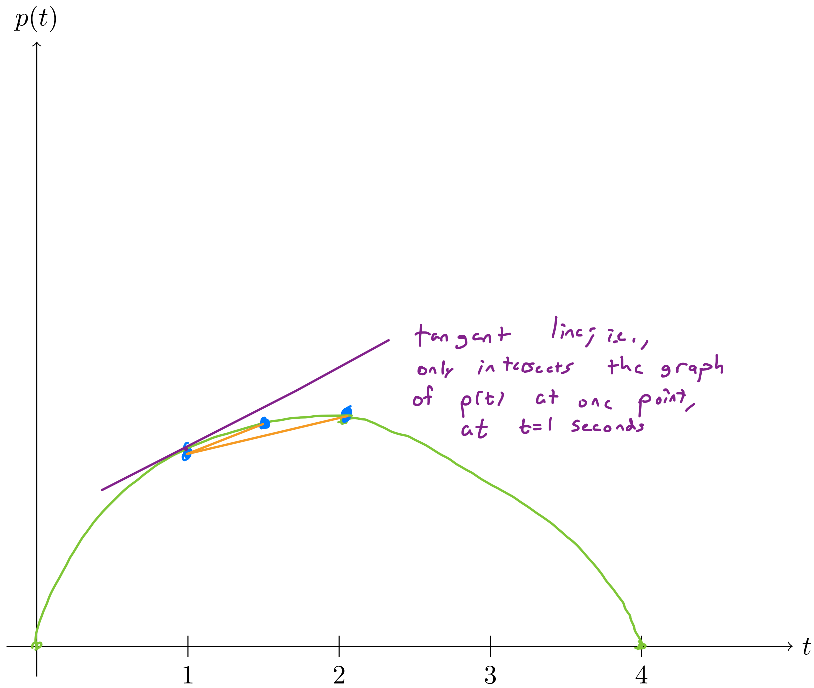
A smaller time window produces an average velocity that is closer to the exact instantaneous velocity at 1s.

Average velocity between $t = 1s$ and $t = 1.001$ seconds:

$$v_{avg} = \frac{\Delta p}{\Delta t} = \frac{p(1.001) - p(1)}{1.001 - 1} \frac{ft}{s} \approx 31.984 \frac{ft}{s}$$

wavy equal sign is "approximately equal."

So we have done several approximations. What is the end goal?



Definition. *Instantaneous Velocity* is the slope of the tangent line at one point of a position fnctn "function"

As the second time t is closer to $t = 1$ in our approximations, the average velocity from $t=1$ sec to t seconds is closer to the instantaneous velocity at $t=1$ sec.

This is a limit! The *limit* as t approaches 1 of the average velocity gives the instantaneous velocity at $t=1$ second

shorter notation:

$$\lim_{t \to 1} v_{avg} = v_{instantaneous}(1)$$

$$\Rightarrow \boxed{\lim_{t \to 1}} \boxed{\frac{p(t) - p(1)}{t-1}} \boxed{=} \boxed{v_{inst.}(1)}$$

"The limit as t approaches 1" "of the average velocity (given by $\frac{p(t)-p(1)}{t-1}$)" "is" "the instaneous velocity at $t=1$."

For our example, we estimate $v_{inst.}(1) = 32 \frac{ft}{s}$

2.2 The Limit of a Function

Learning Objectives: After completing this section, we should be able to

- define the limit of a function and make educated guesses at limits.
- define the one-sided limit of a function and make educated guesses at limits.

2.2.1 Limit Definition

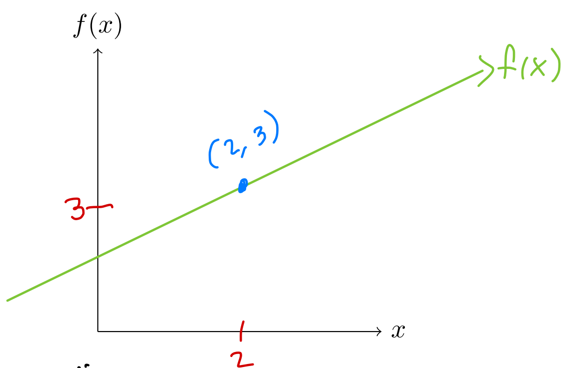
Definition. $\lim_{x \to a} f(x) = L$ means "Limit as x approaches a of f(x) is L."

f(x) is arbitrarily close to L (output of f)

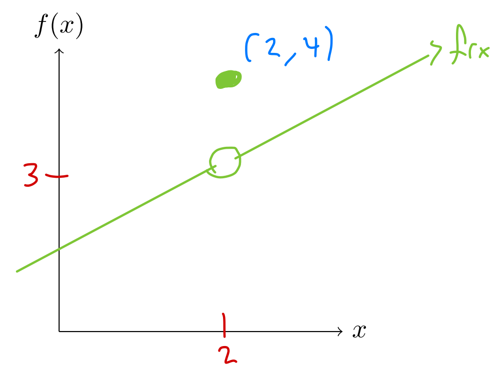
for all x sufficiently close to a (input of f)

The limit is a y-value $\Rightarrow$ L is a number representing a y-value

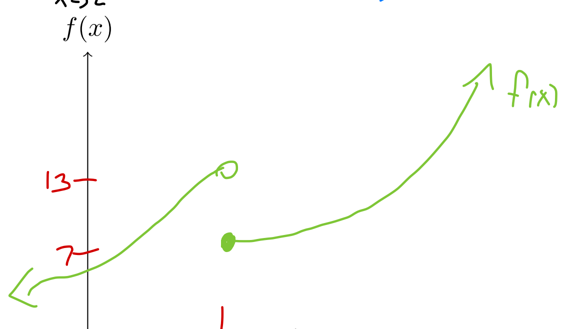
Let's look at several examples:



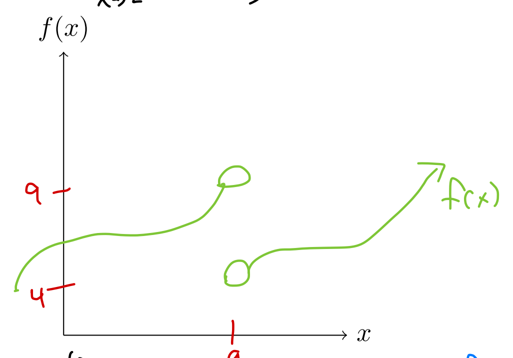
$\lim_{x \to 2} f(x) = 3$; $f(2) = 3$



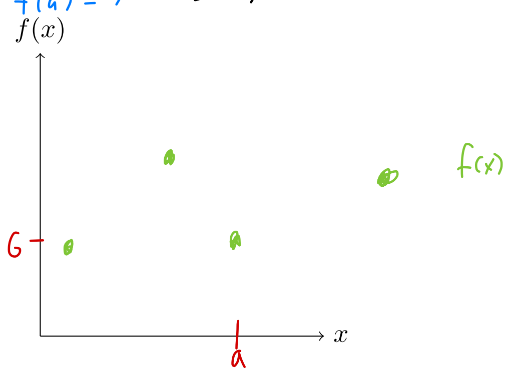
$\lim_{x \to 2} f(x) = 3$; $f(2) = 4$



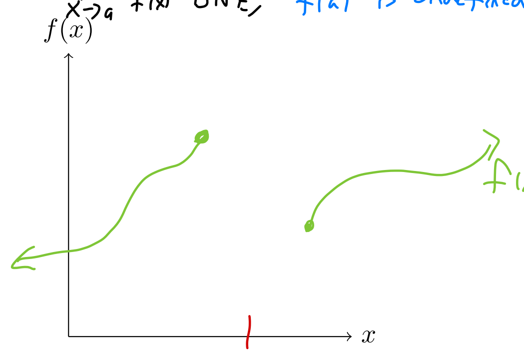
$f(a) = 7$; $\lim_{x \to a} f(x)$ DNE "does not exist"



$\lim_{x \to a} f(x)$ DNE ; $f(a)$ is undefined



$\lim_{x \to a} f(x)$ DNE ; $f(a) = 6$



$\lim_{x \to a} f(x)$ DNE ; $f(a)$ is undefined

Note, for $\lim_{x\to a} f(x) = L$, $f(x)$ must be arbitrarily close to L for all x sufficiently close to a (on both sides of a; i.e., to the left and to the right)

Definition. *(One sided limit)* If $f(x)$ is arbitrary close to L for all x sufficiently close to a, on (only) one side of a.

Definition. *(Right-hand limit)* (only look at $x > a$; i.e., to the right of a)

$$\lim_{x\to a^{+}} f(x) = L$$

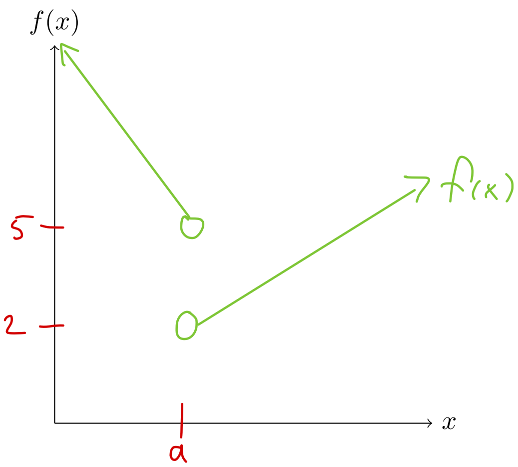
"The limit as x approaches a from the right of $f(x)$ is L."

Definition. *(Left-hand limit)* (only look at $x < a$; i.e., to the left of a)

$$\lim_{x\to a^{-}} f(x) = L$$

"The limit as x approaches a from the left of $f(x)$ is L."

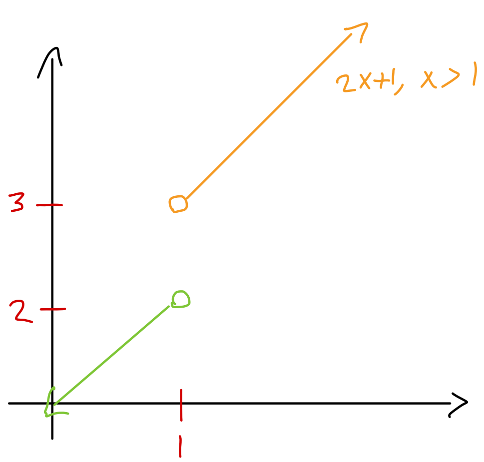
Example.



- $f(a)$ is undefined (no output, as open circles)
- $\lim_{x\to a^-} f(x) = 5$ (as x approaches a from the left, the function output approaches 5)
- $\lim_{x\to a^+} f(x) = 2$ (as x approaches a from the right, the function output approaches 2)
- $\lim_{x\to a} f(x)$ DNE (left & right limits are not the same)

Example. Let

$$f(x) = \begin{cases} 2x+1, & x > 1, \\ 2x, & x < 1. \end{cases}$$



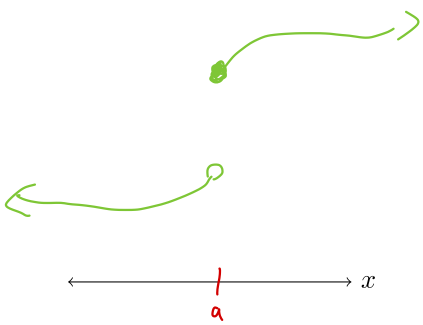
- $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2\cdot 1 = 2$ (only considers $x < 1$)
- $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+1) = 2\cdot 1 + 1 = 3$ (only considers $x > 1$)
- $\lim_{x\to 1} f(x)$ DNE, as $\lim_{x\to 1^-} f(x) = 2$ & $\lim_{x\to 1^+} f(x) = 3$, and $2 \neq 3$
- $f(1)$ is undefined (no output for input 1)

2.2.2 Indeterminate Forms

Question. *What could happen for a function $f(x)$ to **NOT** have a limit?*

Example.

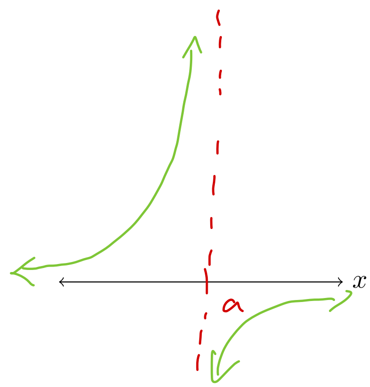
$f(x)$



$\lim_{x\to a} f(x)$ DNE

• piecewise fnctn, but the pieces do not match y-coordinates at the input a

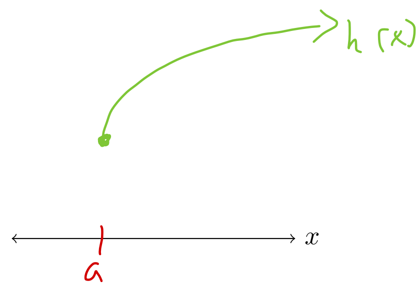
$g(x)$



$\lim_{x\to a} g(x)$ DNE

• vertical asymptote at $x=a$

$h(x)$



$\lim_{x\to a} h(x)$ DNE

• h(x) is undefined on one side of a

Question. *How can we recognize these examples from the functions $f(x)$, $g(x)$, and $h(x)$?*

(In other words, what is the formula doing?)

- f(x) is piecewise, so we check the end of each piece to see if they match (last example on previous page)
- check if we try to divide by 0
- check for negative inputs into even powered roots, e.g., avoid $\sqrt{-1}$ or $\sqrt[4]{-3}$
- logarithms
- piecewise, with no piece defined

In general, if $f(x)$ has *bad behavior* at $x = a$, then $\lim_{x\to a} f(x)$ may not exist.

- divide by 0 at $x=a$
- negative numbers in even powered roots (think $\sqrt{-1}$)
- negative numbers inside logarthims (think $\ln(-3)$)
- pieces do not match at $x=a$
- other strange things (won't be dealt with in MTH 150)

Google "topologists sine curve"

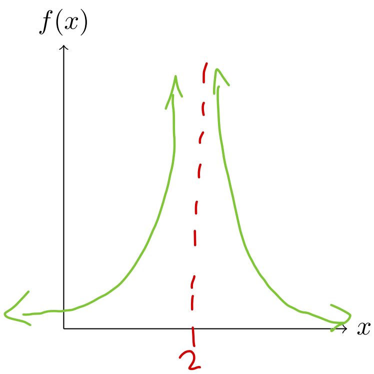
2.2.3 Infinite Limits and Vertical Asymptotes

What does it mean for $\lim_{x\to a} f(x) = \infty$?

"f(x) grows (without bound) (arbitrarily large) (larger than any number) as x gets sufficiently close to a."

$\lim_{x\to a} f(x) = \infty$ means $\lim_{x\to a} f(x)$ DNE

Example. $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$



- When $x=2$, we try to divide by 0; i.e., $\frac{1}{(x-2)^2}$ is not defined at $x=2$
- There is no way to eliminate the bad behavior.
- "$\frac{1}{(x-2)^2}$ grows larger than any number as x approaches the input 2."

If $\lim_{x\to a} f(x) = \pm\infty$ or

$\lim_{x\to a^-} f(x) = \pm\infty$ or

$\lim_{x\to a^+} f(x) = \pm\infty$, then $x=a$ is a vertical asymptote of $f(x)$.

(We only need to approach an infinity from one side to have a vertical asymptote.)

Example. Find all vertical asymptotes of $f(x) = \frac{8x+16}{x^2-4}$.

We have bad behavior if $x^2-4=0 \Rightarrow x^2-4=(x-2)(x+2)=0$

$\Rightarrow x=-2$ and $x=2$ are candidates for vertical asymptotes

First consider $x=2$:

$$\lim_{x\to 2} \frac{8x+16}{x^2-4} \to \frac{8(2)+16}{(2)^2-4} \to \frac{32}{0} \Leftarrow \text{bad behavior!}$$

When evaluating f at $a=2$, we got the form of $\frac{\text{non zero \#}}{0}$.
This is always implies $x=a=2$ is a vertical asymptote.

Let's consider values of x close to 2.

x	2.1	2.001	2.00001
$f(x)=\frac{8x+16}{x^2-4}$	$f(2.1)=\frac{8\cdot 2.1+16}{2.1^2-4}=80$	$f(2.001)=8000$	$f(2.00001)=800{,}000$

Example continued.

If $x \to 2^+$ (x approaches 2 from the right), then $f(x)$ is growing without bound, thus $\lim_{x \to 2^+} f(x) = +\infty$

$\Rightarrow$ Therefore $x = 2$ is a vertical asymptote

$\Rightarrow$ Let's consider $x \to 2^-$

x	1.9	1.999	1.99999
$f(x)$	$f(1.9) = -80$	$f(1.999) = -8000$	$f(1.99999) = -800{,}000$

$\Rightarrow \lim_{x \to 2^-} f(x) = -\infty$, as $f(x)$ is decreasing without bound as x approaches 2 from the left.

Check $x = -2$

$$\lim_{x \to -2} \frac{8x+16}{x^2-4} \to \frac{8(-2)+16}{(-2)^2-4} \to \frac{0}{0}$$

If we get the indeterminate form $\frac{0}{0}$, then the limit can do anything, so we need to more work!

Note $$\lim_{x \to -2} \frac{8x+16}{x^2-4} = \lim_{x \to -2} \frac{8\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}$$

Note $$\frac{8x+16}{x^2-4} = \frac{8(x+2)}{(x+2)(x-2)} \neq \frac{8}{x-2} \quad \text{if } x = -2,$$

as $$\frac{8(-2+2)}{(-2+2)(-2-2)} = \frac{0}{0} \neq \frac{8}{-2-2} = \frac{8}{-4} = -2$$

but $$\frac{8(x+2)}{(x+2)(x-2)} = \frac{8}{x-2} \quad \text{if } x \neq -2$$

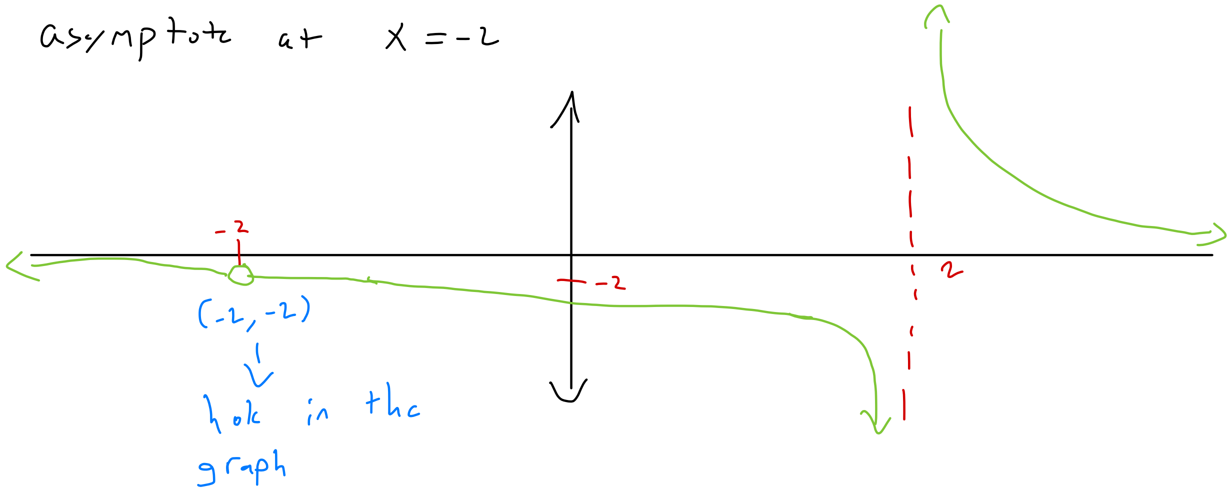
However, $\lim_{x \to -2} \frac{8(x+2)}{(x+2)(x-2)} = \lim_{x \to -2} \frac{8}{x-2}$, as the limit does not care what happens specifically at $x = -2$, only inputs close to -2

So finally

$$\lim_{x \to -2} \frac{8x+16}{x^2-4} = \lim_{x \to -2} \frac{8(x+2)}{(x+2)(x-2)} = \lim_{x \to -2} \frac{8}{x-2} = \frac{8}{-2-2} = \frac{8}{-4} = -2$$

Example continued.

Therefore, $\lim_{x \to -2} f(x)$ exists and is -2. There is no vertical asymptote at $x = -2$



2.3 Calculating Limits

Learning Objectives: After completing this section, we should be able to

- calculate limits using various Limit Laws and properties.

2.3.1 Limit Laws

Suppose that c is any constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist; i.e., they are equal to a real number. Then

1. Constant multiples:

$$\lim_{x\to a}\left(c\cdot f(x)\right) = c\cdot\left(\lim_{x\to a} f(x)\right)$$

2. Sums:

$$\lim_{x\to a}\left(f(x)+g(x)\right) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$$

$$\lim_{x\to a}\left(f(x)-g(x)\right) = \lim_{x\to a}\left(f(x)+(-1)g(x)\right) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a}(-1)g(x)\right)$$
$$= \left(\lim_{x\to a} f(x)\right) + (-1)\left(\lim_{x\to a} g(x)\right) = \left(\lim_{x\to a} f(x)\right) - \left(\lim_{x\to a} g(x)\right)$$

3. Products:

$$\lim_{x\to a}\left(f(x)\cdot g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

4. Quotients:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} g(x)\right)}, \quad \text{so long as } \lim_{x\to a} g(x) \neq 0$$

5. Powers:

$$\lim_{x\to a}\left(f(x)\right)^c = \left(\lim_{x\to a} f(x)\right)^c$$

6. Roots:

$$\lim_{x\to a}\sqrt[c]{f(x)} = \sqrt[c]{\lim_{x\to a} f(x)}$$

For all of these, $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ need to exist!

Example. Suppose $\lim_{x\to a} f(x) = 2$ and $\lim_{x\to a} g(x) = -1$. Compute

$$\lim_{x\to a}\left(5\frac{f(x)}{g(x)} - \left(g\left(x\right)\right)^4 + g(x)\sqrt{f(x)}\right).$$

$$= \lim_{x\to a}\left(5\frac{f(x)}{g(x)}\right) - \lim_{x\to a}\left(g(x)\right)^4 + \lim_{x\to a} g(x)\sqrt{f(x)} \qquad (2)$$

$$= 5\left(\lim_{x\to a}\frac{f(x)}{g(x)}\right) - \left(\lim_{x\to a} g(x)\right)^4 + \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a}\sqrt{f(x)}\right) \qquad (1),\ (5),\ (3)$$

$$= 5\left(\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\right) - \left(\lim_{x\to a} g(x)\right)^4 + \left(\lim_{x\to a} g(x)\right)\left(\sqrt{\lim_{x\to a} f(x)}\right) \qquad (4),\ (6)$$

Since $\lim_{x\to a} f(x) = 2$ and $\lim_{x\to a} g(x) = -1$

$$= 5\left(\frac{2}{-1}\right) - (-1)^4 + (-1)\sqrt{2}$$

2.3.2 Computing Limits

Given $f(x)$, how do we compute limits?

- If there is no bad behavior, just plug in $x = a$.

Example.

$$\lim_{x\to 3}(2x^2+x+4) = 2(3)^2+(3)+4$$

(okay to stop)

$$= 2\cdot 9 + 7 = 18+7 = 25$$

$\Rightarrow$ Got a number, so we're done

- If there is bad behavior, attempt to tame it.

Example. $\lim_{x\to 2}\frac{x^2-4}{x-2} \to \frac{2^2-4}{2-2} \to \frac{0}{0}$ indeterminate!

Try to simplify: $= \lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x\to 2}(x+2)$

$$= 2+2 = 4$$

$$\Rightarrow \lim_{x\to 2}\frac{x^2-4}{x-2} = 4$$

Example. $\lim_{x\to 0}\frac{\frac{1}{5+x}-\frac{1}{5}}{x} \to \frac{\frac{1}{5+0}-\frac{1}{5}}{0} \to \frac{\frac{1}{5}-\frac{1}{5}}{0} \to \frac{0}{0}$ indeterminate

$$= \lim_{x\to 0}\frac{\left(\frac{1}{5+x}-\frac{1}{5}\right)}{x}\frac{(5+x)}{(5+x)} = \lim_{x\to 0}\frac{1-\frac{5+x}{5}}{x(5+x)}$$

Did not quite clean it up, we'll try something else

$\Rightarrow$ Find a common denominator in the numerator

$$= \lim_{x\to 0}\frac{\frac{1}{5+x}\left(\frac{5}{5}\right)-\frac{1}{5}\left(\frac{5+x}{5+x}\right)}{x} = \lim_{x\to 0}\frac{\frac{5}{5(5+x)}-\frac{(5+x)}{5(5+x)}}{x}$$

Common!

✱ Note, $\frac{5+x}{5+x}=1$, $\frac{5}{5}=1$, so we didn't change the problem.

$$= \lim_{x\to 0}\frac{\frac{5-(5+x)}{5(5+x)}}{x} = \lim_{x\to 0}\frac{\frac{\cancel{5}-\cancel{5}-x}{5(5+x)}}{x}$$

$$= \lim_{x\to 0}\frac{\frac{-x}{5(5+x)}}{x} = \lim_{x\to 0}\frac{\frac{(-1)\cancel{(x)}}{5(5+x)}}{\left(\frac{\cancel{x}}{1}\right)}\frac{\left(\frac{1}{\cancel{x}}\right)}{\left(\frac{1}{\cancel{x}}\right)}$$

✱ Note $\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}=1$, so we didn't change the problem

$$= \lim_{x\to 0}\frac{\frac{(-1)}{5(5+x)}}{\left(\frac{1}{1}\right)} = \lim_{x\to 0}\frac{-1}{5(5+x)}$$

$$= \frac{-1}{5(5+0)} = \frac{-1}{25}$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{1}{5+x}-\frac{1}{5}}{x} = -\frac{1}{25}$$

Example. $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$ $\to \frac{\sqrt{0^2+100}-10}{0^2} \to \frac{\sqrt{100}-10}{0} \to \frac{10-10}{0} \to \frac{0}{0}$

Indeterminate

Multiply by the conjugate:

$$= \lim_{x\to 0}\left(\frac{\sqrt{x^2+100}-10}{x^2}\right)\left(\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}\right)$$

fancy 1!

$\sqrt{x^2+100}-10 \xRightarrow{\text{conjugate}} \sqrt{x^2+100}+10$

In general, conjugate for

$\sqrt{a}+b \Leftrightarrow \sqrt{a}-b$

or $a-\sqrt{b} \Leftrightarrow a+\sqrt{b}$

$$= \lim_{x\to 0} \frac{\left(\sqrt{x^2+100}\,\sqrt{x^2+100}\right)+(-10)\left(\sqrt{x^2+100}\right)+\left(\sqrt{x^2+100}\right)(10)+(-10)(10)}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{x^2+100}\right)^2 \cancel{-10\sqrt{x^2+100}} + \cancel{10\sqrt{x^2+100}} - 100}{x^2\left(\sqrt{x^2+100}+10\right)} = \lim_{x\to 0}\frac{\left(\sqrt{x^2+100}\right)^2-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

Note $\left(\sqrt{x^2+100}\right)^2 = \left[(x^2+100)^{\cancel{1/2}}\right]^{\cancel{2}} = x^2+100$

$$= \lim_{x\to 0}\frac{x^2+\cancel{100}-\cancel{100}}{x^2\left(\sqrt{x^2+100}+10\right)} = \lim_{x\to 0}\frac{\cancel{x^2}}{\cancel{x^2}\left(\sqrt{x^2+100}+10\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0^2+100}+10} = \frac{1}{\sqrt{100}+10} = \frac{1}{10+10} = \frac{1}{20}$$

You try!

Example. $\lim_{x\to 3} \frac{\frac{1}{7}+\frac{1}{x-10}}{x-3}$ $\to \frac{\frac{1}{7}+\frac{1}{3-10}}{3-3} \to \frac{\frac{1}{7}+\frac{1}{-7}}{0} \to \frac{0}{0}$ indeterminate

$$\lim_{x\to 3}\frac{\frac{1}{\boxed{7}}\cdot\frac{x-10}{x-10}+\frac{1}{x-10}\left(\frac{7}{7}\right)}{x-3} = \lim_{x\to 3}\frac{\frac{x-10}{7(x-10)}+\frac{7}{7(x-10)}}{x-3}$$

$$= \lim_{x\to 3}\frac{\frac{x-10+7}{7(x-10)}}{x-3}$$

$$= \lim_{x\to 3}\frac{\frac{\cancel{x-3}}{7(x-10)}}{\underset{1}{\cancel{x-3}}}\;\frac{\left(\frac{1}{\cancel{x-3}}\right)}{\left(\frac{1}{\cancel{x-3}}\right)}$$

$$= \lim_{x\to 3}\frac{\frac{1}{7(x-10)}}{\frac{1}{1}} = \frac{1}{7(3-10)} = \frac{1}{7(-7)}$$

$$= -\frac{1}{49}$$

Check: Did you write $\lim_{x\to 3}$ every time until you substituted?

Summary: How to compute $\lim_{x \to a} f(x)$

1. If no bad behavior at $x = a$,

   then $\lim_{x \to a} f(x) = f(a)$ (just plug in $x = a$ at the start)

2. If bad behavior, i.e., $f(a)$ is not a number and is something like $\frac{0}{0}$, then try to tame it by simplifying.

   (a) If $f(x) = \frac{\text{polynomial}}{\text{polynomial}}$, then factor and cancel.

   (b) If $f(x) = \frac{\text{fraction} \pm \text{fraction}}{\text{polynomial}}$, then combine fractions with a common denominator, and then cancel.

   (c) If $f(x) = \frac{\sqrt{\text{something}} \pm \text{something}}{\text{polynomial}}$, then multiply by the conjugate in the numerator and denominator, then simplify and cancel. (The conjugate swaps '+' with '−' or vice versa)

3. If piecewise, and $f(x)$ changes pieces at $x=a$, then consider each piece separately.

   (a) If the pieces match at $x=a$, then that is limit

   (b) If the pieces do not match at $x=a$; i.e., they are different numbers, then the limit Does Not Exist (DNE)

4. If bad behavior cannot be eliminated by using this procedure; i.e., evaluating your simplified fnctn yields $\frac{\text{nonzero number}}{\text{zero}}$, then the limit DNE

2.5 Continuity and the Intermediate Value Theorem

Learning Objectives: After completing this section, we should be able to

- define continuity and discontinuity.
- state and apply the Intermediate Value Theorem.

Definition. A function f is *continuous at* $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

This means 3 things:

1. $\lim_{x\to a} f(x)$ exists; i.e., $\lim_{x\to a^-} f(x) = \lim_{x \to a^+} f(x)$, and it is a number
2. $f(a)$ is defined; i.e., f has a numerical output for the input a.
3. The limit is equal to the function's output at a; i.e., (1) and (2) give the same number.

If any one of these fail, then the fnctn f(x) is discontinuous at X=a.

Example. Consider $f(x) = \frac{x+1}{x^2-4}$.

Note, there is bad behavior if $x^2-4=0 \Rightarrow x^2-4 = (x-2)(x+2) = 0$ when $x=-2$ & 2

So, if $x=-2$ or $x=2$, then f(x) has no output as we try to divide by 0. So, f(x) is discontinuous at $x=-2$ & $x=2$, as it violates criteria (2) from above.

All other inputs (x-values) are well-behaved, so $\lim_{x\to a} \frac{x+1}{x^2-4} = \frac{a+1}{a^2-4} = f(a)$, if $a \neq -2$ or 2.

Thus, f is continuous for all x excluding $x=-2$ or $x=2$
$\Rightarrow$ f is continuous on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

Example. Consider $f(x) = \sqrt{x}$.

If $x<0$, then there is bad behavior as we have negatives inside an even powered root. $\Rightarrow f(x)=\sqrt{x}$ is undefined; i.e., has no output, if $x<0$. So f is discontinuous for $x<0$ by criteria (2).

If $x>0$, then there are no problems, as $\lim_{x\to a} f(x) = \lim_{x\to a}\sqrt{x} = \sqrt{a} = f(a)$, if $a>0$.

Therefore, f is continuous for $x>0$.

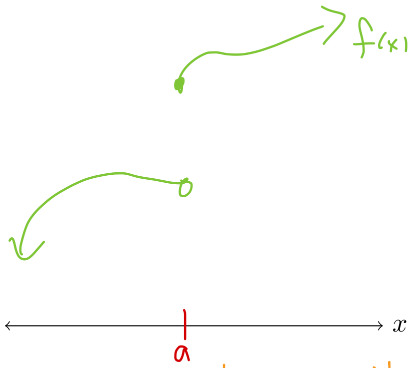
Consider $x=0$. What is $\lim_{x\to 0} f(x) = \lim_{x\to 0}\sqrt{x}$? Consider $\lim_{x\to 0^-}\sqrt{x}$, we are only consider x extremely close to 0, but to the left of 0; i.e., negative x's. $\Rightarrow \lim_{x\to 0^-}\sqrt{x}$ DNE, as $\sqrt{x}$ is undefined for $x<0$. Thus, $\lim_{x\to 0}\sqrt{x}$ DNE, so f is discontinuous at $x=0$ via criteria (1). So f is continuous for $x>0$.

However, $\lim_{x\to 0^+}\sqrt{x} = 0 = \sqrt{0}$, so $\lim_{x\to 0^+} f(x) = f(0)$. This implies f is continuous from the right at $x=0$.

Note, a fnctn f is continuous from the left at $x=a$ if $\lim_{x \to a^-} f(x) = f(a)$.

Types of discontinuities:

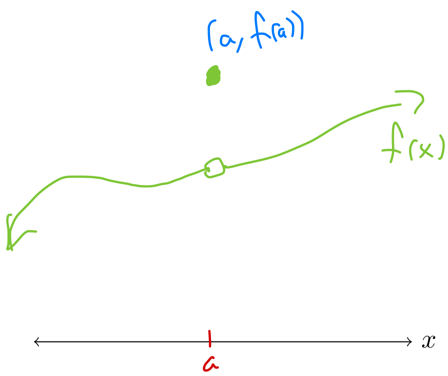
- Jump



$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$,

so $\lim_{x \to a} f(x)$ DNE, so f is discontinuous at $x=a$.

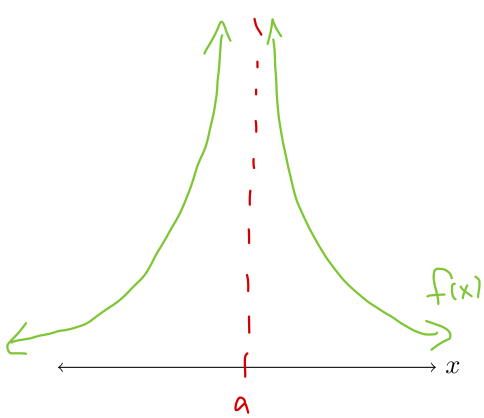
- Removable "hole in the graph"



- $\lim_{x \to a} f(x)$ does exist
- $f(a)$ exists
- $\lim_{x \to a} f(x) \neq f(a)$

so f is discontinuous at $x=a$.

- Infinite



- Note $\lim_{x \to a^-} f(x) = +\infty$ and $\lim_{x \to a^+} f(x) = +\infty$

  so $\lim_{x \to a} f(x) = +\infty$

  Recall, ∞ is not a number, so technically, $\lim_{x \to a} f(x)$ DNE

  $\Rightarrow$ discontinuous

- Also, $f(a)$ is undefined, so definitely discontinuous

$\Rightarrow$ f is discontinuous at $x=a$.

Example. Let $f(x) = \begin{cases} x^2 - c, & \text{if } x < 5, \\ 4x + 2c, & \text{if } x \geq 5. \end{cases}$

Find c such that f is continuous.

Note $x^2 - c$ and $4x + 2c$ both have no bad behavior, so the only potential problem is when pieces switch at $x = 5$

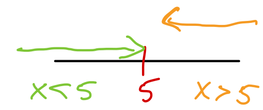
By defn ("definition"), f is cts ("continuous") at $x=5$ if

$$\lim_{x \to 5} f(x) = f(5).$$

To make $\lim_{x\to 5} f(x)$ exist, we need left & right limits to agree:

$\lim_{x \to 5^-} f(x) = \lim_{x\to 5^-} x^2 - c = 5^2 - c = 25 - c$

considers $x < 5$



$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} 4x + 2c = 4 \cdot 5 + 2c = 20 + 2c$

considers $x > 5$

(★ space out better)

Note $\lim_{x\to 5} f(x)$ exists if $\lim_{x \to 5^-} f(x) = \lim_{x\to 5^+} f(x)$

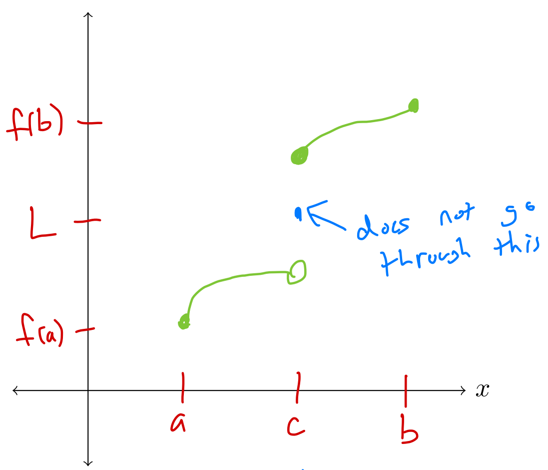
$\Rightarrow 25 - c = 20 + 2c$
$\quad -20+c \quad -20+c$

$\Rightarrow 5 = 3c \Rightarrow c = \frac{5}{3}$

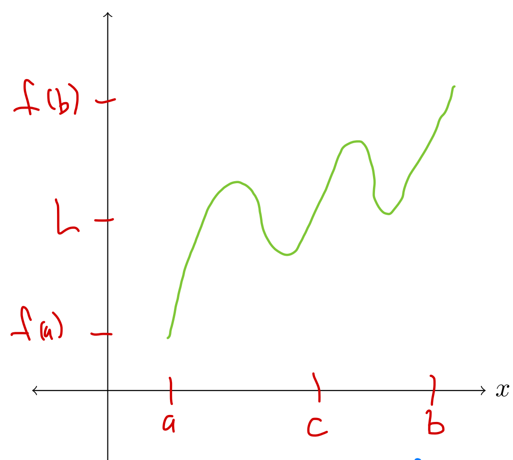
So, if $c = \frac{5}{3}$, then $\lim_{x\to 5^-} f(x) = 25 - \frac{5}{3} = 20 + 2\cdot\frac{5}{3} = \lim_{x \to 5^+} f(x) = \lim_{x\to 5} f(x) = 4\cdot 5 + 2\left(\frac{5}{3}\right)$

$\Rightarrow f$ is continuous everywhere if $c = \frac{5}{3}$

Question. *True or False: Pick any number L between $f(a)$ and $f(b)$. Then, there is an x-value c between a and b such that $f(c) = L$.*



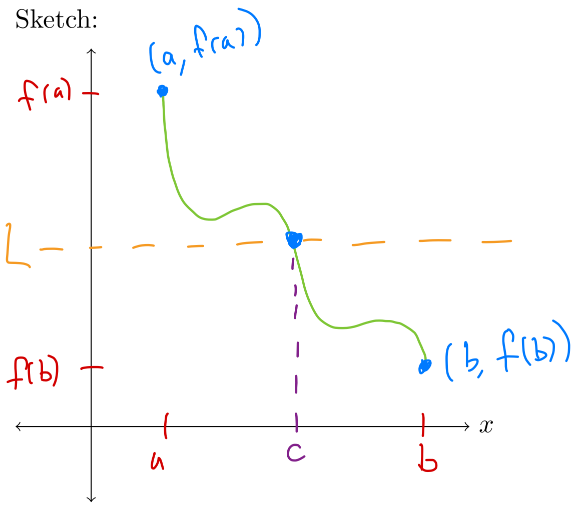
False!



True, if f is continuous

Theorem (Intermediate Value Theorem). *Assume f is continuous on $[a, b]$, and* L is any number between $f(a)$ and $f(b)$. Then there exists a c in (a, b) such that $f(c) = L$.

Sketch:



Challenged with the y-value L between $f(a)$ & $f(b)$, we found an input c such that $f(c) = L$, The Intermediate Value Theorem (IVT) guarantees such a c exists between a & b.

Translation: A continuous functn $f(x)$ hits all intermediate values (outputs) on the interval from (a, b).

Why do we care about IVT? On its own, the IVT is not particularily useful, though are some applications

(1) The IVT helps prove the Mean Value Theorem, is extremely useful.

(2) From there, the Mean Value Theorem helps prove the Fundamental Theorem of Calculus (is extremely useful!)

Application of IVT: Root finding problems

Example. Kepler's equation for orbits (planets, satellites, etc...) is given by $y = x - a\sin(x)$ where a is a constant appropriate for the problem.

Suppose we measure $y = 1.4$ and $a = 0.1$.

$$y = x - a \cdot \sin(x)$$
$$1.4 = x - 0.1 \cdot \sin(x).$$

Can we find x so that this equation is true?

This is impossible to solve using algebra/trig

$\Rightarrow$ We'll convert this to a root finding problem.

$$1.4 = x - 0.1\sin(x) \Rightarrow 0 = x - 0.1\sin(x) - 1.4$$

Example continued.

Call $f(x) = x - 0.1\sin(x) - 1.4$. Then if we find x such that $f(x) = 0$, then we have solved the equation.

We'll use the IVT to show a root/solution exists between 0 and π.

Note, $f(x) = x - 0.1\sin(x) - 1.4$ is continuous on the interval $[0, \pi]$ (a = 0, b = π)

Also, $f(a) = f(0) = 0 - 0.1 \cdot \sin(0) - 1.4 = -1.4$

and $f(b) = f(\pi) = \pi - 0.1 \cdot \sin(\pi) - 1.4 \approx 1.7$

Since $L = 0$ is between $f(0) = -1.4$ and $f(\pi) \approx 1.7$, the IVT guarantees there is a c between 0 & π such that $f(c) = 0$.

The IVT only says a solution exists, but does not give an algorithm to find it.

$\Rightarrow$ (approximate answer $x \approx 1.49975\ldots$ not our course to find it, i.e., $f(1.49975) \approx 0$)

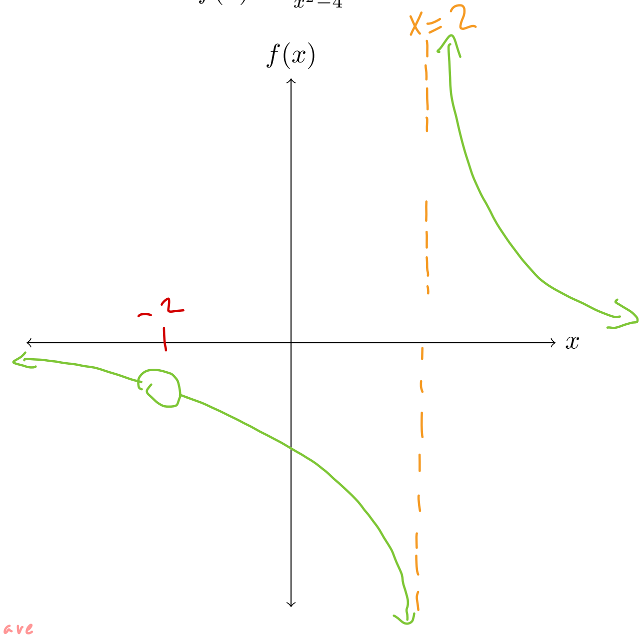
IVT doesn't provide a solution, it only tells us one exists

2.6 Limits at Infinity and Horizontal Asymptotes

Learning Objectives: After completing this section, we should be able to

- define the limits of a function at infinity and determine horizontal asymptotes of functions, if there are any.
- understand the infinite limits of a function at infinity.

Example. We've encountered the function $f(x) = \frac{8x+16}{x^2-4}$ before.



It looks like it may have ~~maybe~~ a horizontal asymptote too. Perhaps $y = 0$?

Definition. $x = a$ is a *vertical asymptote* if

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$ (from before)

Definition. $y = L$ is a *horizontal asymptote* if

$$\lim_{x \to \pm\infty} f(x) = L$$

Example.

$$\lim_{x\to\infty} \frac{8x+16}{x^2-4}$$

The numerator $8x+16 \to \infty$ as $x \to \infty$
The denominator $x^2-4 \to \infty$ as $x \to \infty$
$\Rightarrow \lim_{x\to\infty} \frac{8x+16}{x^2-4} \to \frac{\infty}{\infty}$

What is $\frac{\infty}{\infty}$? It is an indeterminate value. when both the numerator and denominator each approach infinity, then the overall limit could be anything (a number, an infinity, DNE, ..)

Can we do some algebra to clean up $\lim_{x\to\infty} \dfrac{8x+16}{x^2-4}$ and get an actual value instead of an indeterminate form?

$$\lim_{x\to\infty} \frac{8x+16}{x^{2}-4}\cdot\frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{8x\left(\frac{1}{x^2}\right)+16\left(\frac{1}{x^2}\right)}{x^2\left(\frac{1}{x^2}\right)-4\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{\frac{8}{x}+\frac{16}{x^2}}{1-\frac{4}{x^2}}$$

highest power of x in the problem

"zoom in" on each term

$\lim_{x\to\infty} \dfrac{8}{x}$ $\quad \dfrac{\text{constant numerator}}{\text{denominator grows without bound}}$

x	1	8	32	100,000	big
$\frac{8}{x}$	$\frac{8}{1}=8$	$\frac{8}{8}=1$	$\frac{8}{32}=\frac{1}{4}$	0.00008	close to 0

So $\lim_{x\to\infty} \dfrac{8}{x} = 0$

Similarly, $\lim_{x\to\infty}\dfrac{16}{x^2}=0$ and $\lim_{x\to\infty}\dfrac{4}{x^2}=0$

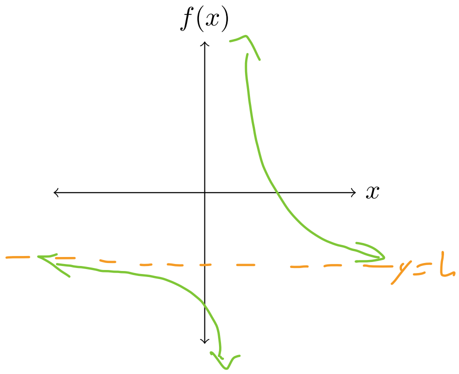
$$\lim_{x\to\infty}\frac{\frac{8}{x}+\frac{16}{x^2}}{1-\frac{4}{x^2}} = \frac{\lim_{x\to\infty}\frac{8}{x}+\lim_{x\to\infty}\frac{16}{x^2}}{\lim_{x\to\infty}1-\lim_{x\to\infty}\frac{4}{x^2}} = \frac{0+0}{1-0} = \frac{0}{1} = 0$$

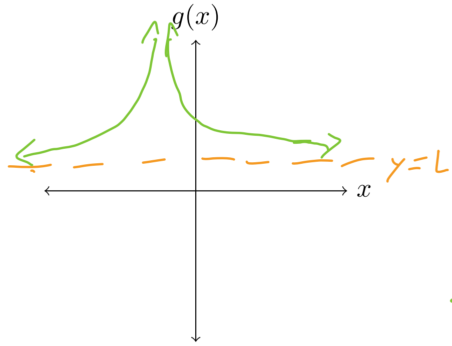
so $\lim_{x\to\infty}\dfrac{8x+16}{x^2-4} = \cdots = 0$

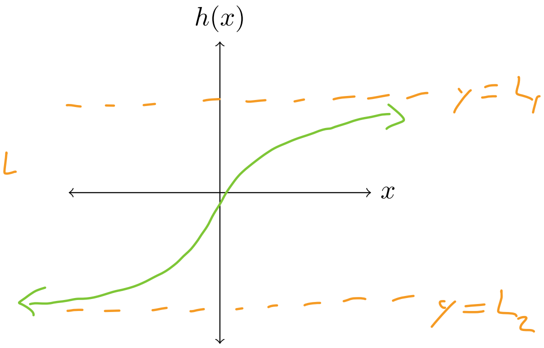
By the defn of horizontal asymptote,

$y=0$ is a horizontal asymptote of $\dfrac{8x+16}{x^2-4}$

Question. *Is it possible to have 2 horizontal asymptotes?*







$\lim_{x\to\infty} f(x) = L$ $\quad\lim_{x\to\infty} g(x) = L$ $\quad\lim_{x\to\infty} h(x) = L_1$

$\lim_{x\to-\infty} f(x) = L$ $\quad\lim_{x\to-\infty} g(x) = L$ $\quad\lim_{x\to-\infty} h(x) = L_2$

Example. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

Strategy: Divide by the highest power of x in the denominator: $\sqrt{x^2}$

Is $\sqrt{x^2} = x$?

$\boxed{\text{Ex}}$ $x=1$: $\sqrt{x^2} = \sqrt{1^2} = \sqrt{1} = 1 = x$ ✓ ; $x=4$: $\sqrt{x^2} = \sqrt{4^2} = \sqrt{16} = 4 = x$ ✓ ; $x=-4$: $\sqrt{x^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq -4$ ✗

$\Rightarrow \sqrt{x^2} \neq x$ for all x

$$\Rightarrow \sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases} \qquad \Rightarrow \sqrt{(-4)^2} = -(-4) = 4 \checkmark$$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \frac{\left(\frac{1}{\sqrt{x^2}}\right)}{\left(\frac{1}{\sqrt{x^2}}\right)} = \lim_{x\to\infty} \frac{\frac{x}{\sqrt{x^2}}}{\sqrt{\frac{x^2+1}{x^2}}}$$

Note, $\sqrt{x^2} = x$ if $x \geq 0$. Our limit is as $x\to\infty$; i.e., $x>0$

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{\lim_{x\to\infty} 1}{\sqrt{\lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0}} = \frac{1}{\sqrt{1}} = 1$$

Since $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1$, we have $y=1$ is a horizontal asymptote

---

Consider $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} \frac{\left(\frac{1}{\sqrt{x^2}}\right)}{\left(\frac{1}{\sqrt{x^2}}\right)} = \lim_{x\to-\infty} \frac{\frac{x}{\sqrt{x^2}}}{\sqrt{\frac{x^2+1}{x^2}}}$

Note, $\sqrt{x^2} = -x$, if $x<0$. Our limit is as $x\to-\infty$; i.e., $x<0$

$$= \lim_{x\to-\infty} \frac{\frac{x}{-x}}{\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to-\infty} \frac{-1}{\sqrt{1+\frac{1}{x^2}}} = \frac{\lim_{x\to-\infty} -1}{\sqrt{\lim_{x\to-\infty} 1 + \lim_{x\to-\infty}\frac{1}{x^2}}} = \frac{-1}{\sqrt{1+0}} = \frac{-1}{\sqrt{1}} = -1$$

Since $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$, we have $y=-1$ is a horizontal asymptote

Question. *Is it possible to have more than 2 horizontal asymptotes?*

To find HA (horizontal asymptote), we consider only two limits

$\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$

No, there cannot be more than 2 HA

Question. *How many vertical asymptotes can we have?*

Any finite amount

$$\frac{1}{x(x-1)(x-2)(x-3)\cdots(x-N)}$$

each cause VA, i.e., $x=0, x=1, x=2, \ldots, x=N$ are aN VA

Consider $\lim_{x\to\infty} \frac{ax^n}{bx^m}$.

- If $n > m$, the limit is

  ∞ or $-\infty$ (depends on a, b, n, m)

- If $n < m$, the limit is

  0 (also works for $\lim_{x\to-\infty} \frac{ax^n}{bx^m}$)

- If $n = m$, the limit is

  $\frac{a}{b}$ (also works for $\lim_{x\to-\infty} \frac{ax^n}{bx^m}$)

Example. $\lim_{x\to\infty} \frac{5x^5 - 6x^2 + 10^{1000}}{3x^5 + 10x^3 - 1} = \frac{5}{3}$, as the powers of x in the numerator and the denominator are the same, so the resulting limit is the ratio of their coefficients.

Note, $y = \frac{5}{3}$ is a horizontal asymptote for $\frac{5x^5 - 6x^2 + 10^{1000}}{3x^5 + 10x^3 - 1}$

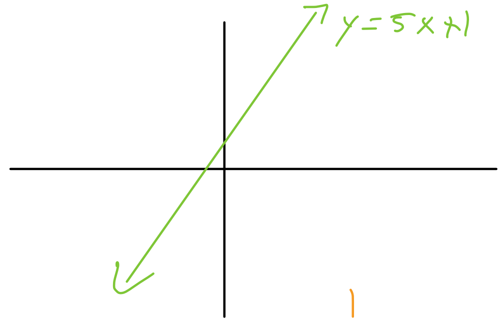
Example. $\lim_{x\to\infty} \frac{10^{100}x^5}{0.001x^{5.01}} = 0$, as the highest power of x in the denominator (5.01) is greater than the highest power of x in the numerator (5)

Note, $y=0$ is a horizontal asymptote for $\frac{10^{100}x^5}{0.001x^{5.01}}$

Fix spacing

Question. *Note that* $\lim_{x\to\infty} 5x+1=\infty$. *Are there any horizontal or vertical asymptotes?*

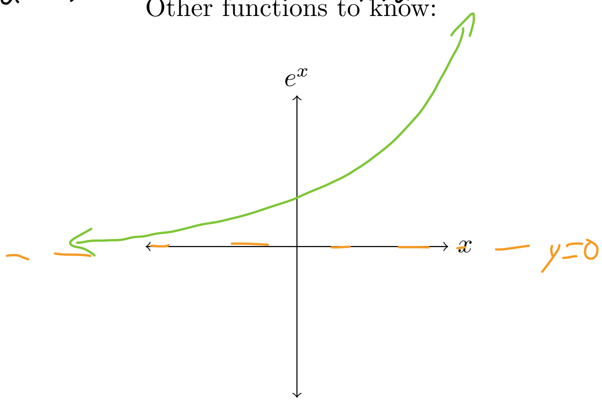
$\lim_{x\to\infty} 5x+1 = \lim_{x\to\infty}\frac{5x^1+1}{x^0} = \infty$, or $-\infty$ as the highest power of x in the numerator (1) is greater than highest power of x in the denominator (0).

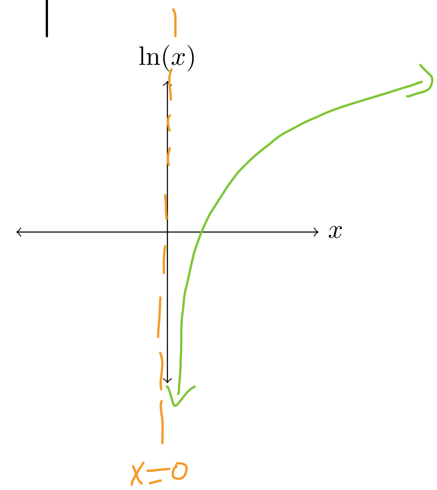


No, as a HA would require $\lim_{x\to\infty} 5x+1 = L$ or $\lim_{x\to-\infty} 5x+1 = L$, for some finite L, which is not the case.

Also, there is no VA, as there ~~are~~ is no inputs a such that $\lim_{x\to a^\pm} 5x+1 = \pm\infty$.

Other functions to know:





e^x has a horizontal asymptote of $y=0$, as $\lim_{x\to-\infty} e^x = 0$

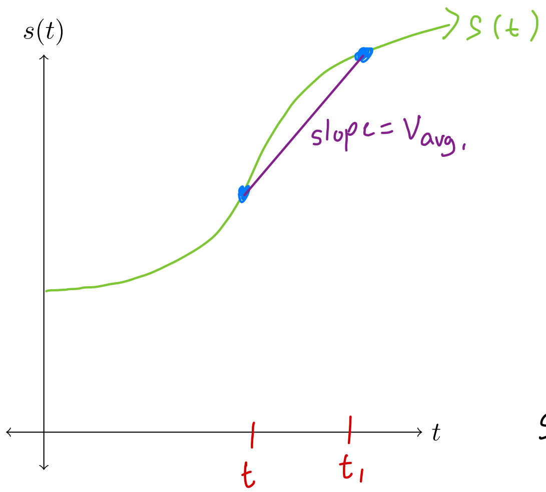
$\ln(x)$ has a vertical asymptote of $x=0$, as $\lim_{x\to 0^+} \ln(x) = -\infty$

2.7 Derivatives and Rates of Change

Learning Objectives: After completing this section, we should be able to

- define the slope of the tangent line to a curve at a point as the limit of the slopes of secant lines of the curve.
- define the instantaneous velocity of a moving object as the limit of its average velocity.
- establish the definition of the derivative and interpret it as the slope of the tangent line to a curve.
- interpret the derivative as the instantaneous rate of change.

Recall from earlier: If we know position $s(t)$, how do we get the instantaneous velocity at time t?



V_{avg} = slope of secant line

$$= \frac{s(t) - s(t_1)}{t - t_1}$$

As $t \to t_1$, $V_{avg} \to V(t_1)$ instantaneous velocity

So, the instantaneous velocity at t_1 is

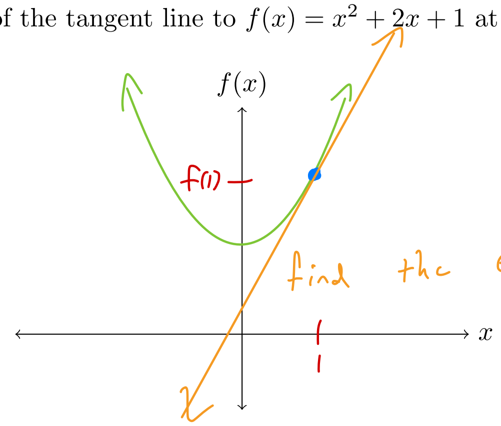
$$V(t_1) = \lim_{t \to t_1} \frac{s(t) - s(t_1)}{t - t_1}$$

Definition. The *instantaneous rate of change* of $f(x)$ at $x = a$ is given by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

- This is the slope of the tangent line to $f(x)$ at $x = a$.
- If f is a position fnctn, then we call instantaneous rate of change the instantaneous velocity.

Example. Find the equation of the tangent line to $f(x) = x^2 + 2x + 1$ at $x = 1$.



find the equation for this line

- To write the equation of any line, we need two things
  (1) slope
  (2) point

Example continued: $f(x) = x^2 + 2x + 1$

$$\text{slope} = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{(x^2+2x+1) - (1^2+2\cdot 1+1)}{x-1}$$

$$= \lim_{x \to 1} \frac{x^2+2x+1-(4)}{x-1} = \lim_{x \to 1} \frac{x^2+2x-3}{x-1} \to \frac{1^2+2\cdot 1-3}{1-1} \to \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{(x+3)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} x+3 = 1+3 = 4.$$

So slope $= m = 4$.

The point at $x=1$ on the graph of $f(x) = x^2+2x+1$ is $(1, f(1))$
$\Rightarrow f(1) = 1^2 + 2\cdot 1 + 1 = 4$, so the point is $(1,4)$

Recall, the point-slope form of a line through (x_1, y_1) with slope m is given by $(y - y_1) = m(x - x_1)$.

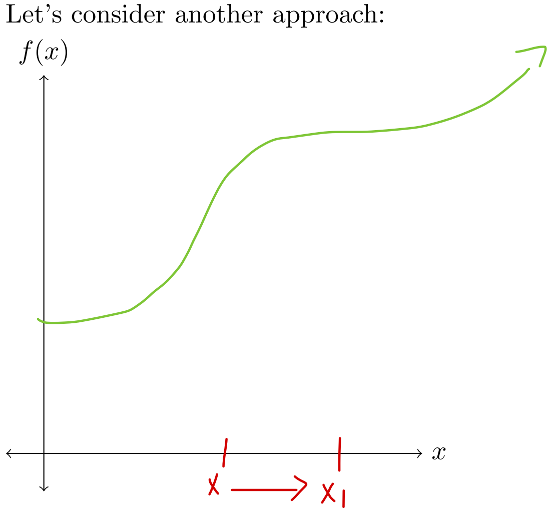
Here, $m = 4$ and $(x_1, y_1) = (1, 4)$, so our tangent is given by

$$(y - 4) = 4\,(x - 1)$$

y-coordinate slope x-coordinate

It is okay to leave it like this for me, but some WebAssign questions want the form $y = \ldots$ $\Rightarrow y = 4(x-1) + 4$

Let's consider another approach:



As $x \to x_1$, the distance between x and x_1 goes to 0, i.e.

$|x - x_1| \to 0$

arrow

Let $x - x_1 = h$, so $x = x_1 + h$

We consider $x \to x_1$

$$\Rightarrow \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{h \to 0} \frac{f(x_1+h) - f(x_1)}{h},$$ as $x \to x_1$, $x - x_1 \to 0$ $\Rightarrow h \to 0$

So, slope of tangent line to f at x_1 $= \displaystyle\lim_{h \to 0} \frac{f(x_1+h) - f(x_1)}{h}$.

Example. Let's find the slope of $f(x) = x^2 + 2x + 1$ at $x = 1$ again with this alternative limit.

$$\text{slope} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[(1+h)^2 + 2(1+h) + 1] - [1^2 + 2 \cdot 1 + 1]}{h}$$

$$= \lim_{h \to 0} \frac{[(h^2+2h+1) + 2 \cdot 1 + 2 \cdot h + 1] - [4]}{h}, \quad \text{as} \quad (1+h)^2 = (1+h)(1+h) = h^2+2h+1$$

and $2(1+h) = 2 \cdot 1 + 2 \cdot h$

$$= \lim_{h \to 0} \frac{h^2 + 2h + 1 + 2 + 2h + 1 - 4}{h} = \lim_{h \to 0} \frac{h^2 + 4h \cancel{+4} \cancel{-4}}{h}$$

$$= \lim_{h \to 0} \frac{h^2+4h}{h} = \lim_{h \to 0} \frac{\cancel{h}(h+4)}{\cancel{h}} = \lim_{h \to 0} h+4$$

$$= 0 + 4 = 4$$

Matches from before!

You try!

Example. Find the equation of the tangent line $f(x) = (x-1)^2$ at $x = 2$.

$$\text{slope} = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} \quad \text{or} \quad \text{slope} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\text{slope} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{[(2+h)-1]^2 - [(2-1)]^2}{h}$$

$$= \lim_{h \to 0} \frac{[1+h]^2 - [1]^2}{h} = \lim_{h \to 0} \frac{[h^2+2h+1] - 1}{h}$$

$$= \lim_{h \to 0} \frac{h^2+2h}{h} = \lim_{h \to 0} \frac{\cancel{h}(h+2)}{\cancel{h}} = \lim_{h \to 0} (h+2)$$

$$= 0 + 2 = 2$$

$$\text{point} = (2, f(2)) = (2, (2-1)^2) = (2, 1)$$

So $(x_1, y_1) = (2, 1)$ and $m = 2$

$$(y - y_1) = m(x - x_1)$$

$$\Rightarrow y - 1 = 2(x-2)$$

$$\Rightarrow y = 2(x-2) + 1$$

(fine for exams)
(use for WeBAssign)

Definition. The *slope of the tangent line* at a point x is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Note, x is a generic variable, and if we compute this limit, then we get a new function that depends on x.

Defn: The derivative of $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

"f prime of x"

provided the limit exists. If the limit exists for an x, then we say f is differentiable at x.

Alternative notation:

~~**Definition.** The *derivative of $f(x)$* is the function~~

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} \qquad \frac{\text{"change in } f\text{"}}{\text{"change in } x\text{"}}$$

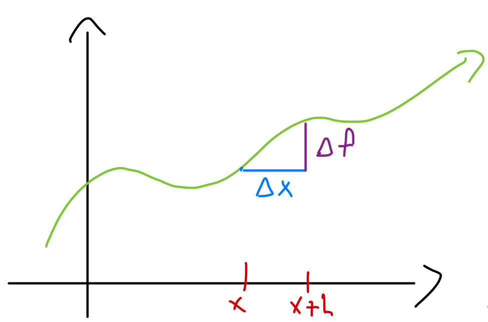
$$= \frac{df}{dx} \Leftarrow \text{derivative of } f \text{ with respect to } x$$

"df divided by dx"

$\Rightarrow$ "df dx" (understood that there is division)

How do we interpret the derivative?

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{slope of secant line through } (x, f(x)) \text{ and } (x+h, f(x+h))}$$



Units: The units of $f'(x)$ is $\dfrac{\text{unit of } f}{\text{unit of } x}$

Ex) If $f(x)$ outputs miles when inputted hours, then $f'(x)$ has a unit of $\frac{\text{miles}}{\text{hour}}$ = mph

Ex) If $g(x)$ is in mph with input of hours, then $g'(x)$ has a unit of $\dfrac{\text{miles}/\text{hour}}{\text{hour}} = \dfrac{\text{miles}}{\text{hour}^2}$

~~What is the derivative? How do we interpret what it means?~~

(acceleration)

Ex) Suppose $h(x)$ is GDP in dollars with input of years, then $h'(x)$ has unit of $\frac{\text{dollars}}{\text{year}} \Rightarrow$ Rate of change of GDP

Summary:

- The derivative at a point $x = a$ is

slope of tangent line at $x = a$ $= f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

- The derivative at any point x is

slope of tangent line at x $= f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

- What does $f'(x)$ mean?
  - Instantaneous rate of change of f with respect to x

equivalent

  - Slope of the tangent line to the curve f at x
  - Slope of f at x
  - If f is a displacement or position function, then $f'(x)$ is velocity
  - In general, unit of $f'(x) = \frac{\text{unit of } f}{\text{unit of } x}$

2.8 Derivative as a Function

Learning Objectives: After completing this section, we should be able to

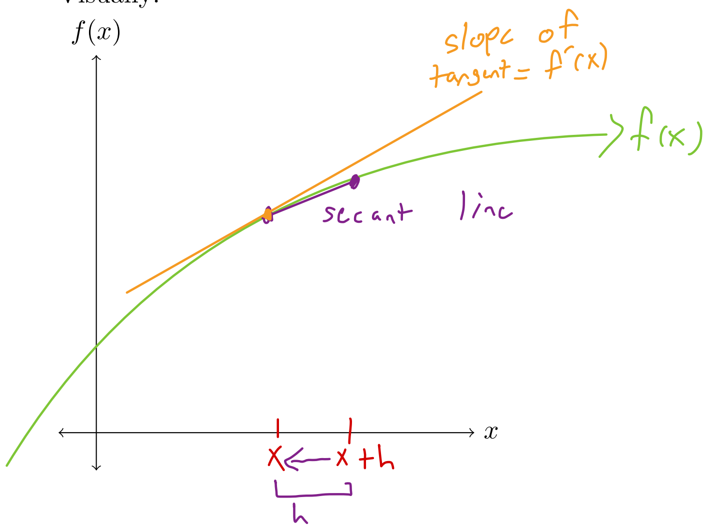
- define and find the derivative f' as a new function derived from a function f.
- denote a derivative using Leibniz notation and prove the fact that the if a function is differentiable then it is continuous.
- analyze the cases in which a function fails to be differentiable.
- analyze whether the derivative of a function is differentiable.

Definition. Recall that the *derivative* of $f(x)$ is given by

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(Note, the verb of taking a "derivative" is differentiate)

Visually:



- As $h \to 0$, the endpoints on the secant line segment "crash" at x
- The limit of the slopes of the secant lines is the derivative.

A common problem is finding the equation of a tangent line to a function. We need

- point $(x, f(x))$
- slope (derivative)

Example. Recall from ~~last time~~ previous section, we found the equation of the tangent line to $f(x) = x^2 + 2x + 1$ at $x = 1$.

point: $(1, f(1)) = (1, 1^2 + 2\cdot 1 + 1) = (1, 4)$

Slope: 4 ~~$f'(1)$, Note $\lim_{h \to 0} f(x+$~~

$$\Rightarrow \quad \underbrace{y - 4 = 4(x-1)}_{\text{tangent line equation}}$$

The tangent line equation is not the same as the derivative fnctn.

Example. What is the derivative of $f(x) = x^2 + 2x + 1$?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(x+h)^2 + 2(x+h) + 1\right] - \left[x^2 + 2x + 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[(\cancel{x^2} + 2hx + h^2) + \cancel{2x} + 2h \cancel{+1}\right] \cancel{-x^2} \cancel{-2x} \cancel{-1}}{h} = \lim_{h \to 0} \frac{2hx + h^2 + 2h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 2)}{\cancel{h}} = \lim_{h \to 0} (2x + h + 2)$$

$$= 2x + 0 + 2 = 2x + 2$$

So, $f'(x) = 2x + 2$.

By the way, $f'(1) = 2 \cdot 1 + 2 = 4$, matching from before!

Example. Find the equation of the tangent line to $f(x) = x^2 + 2x + 1$ whose slope is 6.

For a line, we need a (1) slope
and (2) point

Here, we know the slope, and we need to find the point.

Where is the slope 6?

"slope" "is"

$$f'(x) = 6$$

$$\Rightarrow 2x + 2 = 6 \quad \text{(previous example found } f'(x) = 2x+2\text{)}$$

$$\Rightarrow 2x = 4$$

$$\Rightarrow x = 2$$

point : $(2, f(2)) = (2, 2^2 + 2 \cdot 2 + 1) = (2, 9)$

slope : 6

The equation of the tangent line to f with slope 6 is given by

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \boxed{y - 9 = 6(x - 2)}$$

You try!

Example. Find the derivative of $f(x) = \frac{1}{3x-1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[\frac{1}{3(x+h)-1}\right]\left(\frac{3x-1}{3x-1}\right) - \left[\frac{1}{3x-1}\right]\left(\frac{3(x+h)-1}{3(x+h)-1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(3x-1) - [3(x+h)-1]}{(3(x+h)-1)(3x-1)}}{h} = \lim_{h\to 0} \frac{\frac{3x-1-[3x+3h-1]}{(3(x+h)-1)(3x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{3x}-\cancel{1}-\cancel{3x}-3h+\cancel{1}}{(3(x+h)-1)(3x-1)}}{h} = \lim_{h\to 0}\frac{\frac{-3\cancel{h}}{(3(x+h)-1)(3x-1)}}{\cancel{h}} \quad \frac{\left(\frac{1}{\cancel{h}}\right)}{\left(\frac{1}{\cancel{h}}\right)}$$

$$= \lim_{h\to 0} \frac{-3}{(3(x+h)-1)(3x-1)}$$

$$= \frac{-3}{(3(x+0)-1)(3x-1)} = \frac{-3}{(3x-1)(3x-1)}$$

$$= \frac{-3}{(3x-1)^2}$$

So $\quad f'(x) = \frac{-3}{(3x-1)^2}$

You try!

Example. Find the equation of the tangent line to $f(x) = \frac{1}{3x-1}$ at $x = 1$.

slope: $\quad m = f'(1) = \frac{-3}{(3\cdot 1-1)^2} = \frac{-3}{2^2} = -\frac{3}{4}$

point: $\quad (1, f(1)) = \left(1, \frac{1}{3\cdot 1-1}\right) = \left(1, \frac{1}{3-1}\right) = \left(1, \frac{1}{2}\right)$

Tangent line: $\quad y - y_1 = m(x - x_1)$

$$\Rightarrow y - \frac{1}{2} = -\frac{3}{4}(x-1)$$

2.8.1 Differentiablity

Recall $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, if the limit exists. When does it not exist?

Definition. The derivative of the function $f(x)$ is given by

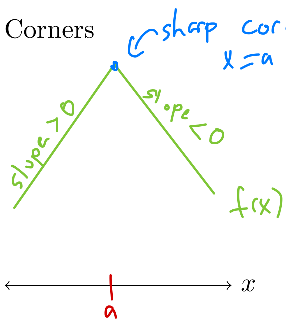
$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{when this limit exists.}$$

Aside: $f'(3)$ is the derivative evaluated at $x=3$. This is equivalent to writing

$\left.\frac{df}{dx}\right|_{x=3}$ $\Leftarrow$ "The derivative of f with respect to x evaluated at $x=3$"

When will we get $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ does not exist?

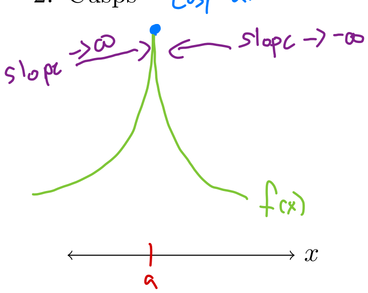
1. Corners



$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \neq \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$,

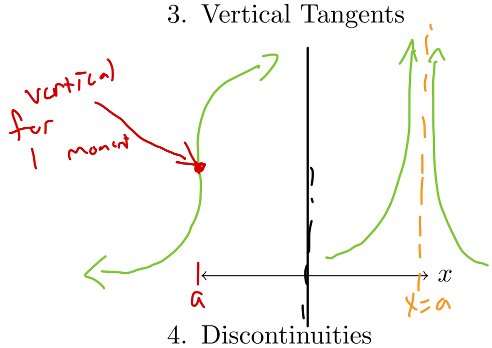
so $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ DNE; i.e., f is not differentiable at $x=a$.

2. Cusps



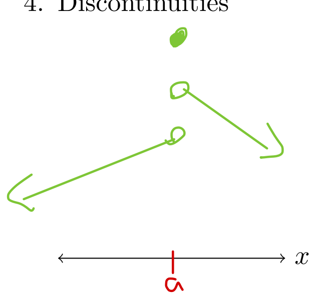
$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ DNE, as left and right limits are different.

3. Vertical Tangents



f is not differentiable at $x=a$

4. Discontinuities



f is not differentiable at $x=a$

Theorem. *If f is differentiable at $x = a$, then*

f is continuous at $x=a$.

Proof. Assume $f(x)$ is differentiable at a.

This means, by definition

$\lim\limits_{h\to 0} \dfrac{f(\overset{a}{\cancel{x}}+h)-f(\overset{a}{\cancel{x}})}{h}$ exists and it is $f'(a)$.

To show f is continuous at a, we need $\lim\limits_{x\to a} f(x) = f(a)$.

Note $\underset{\substack{\uparrow \\ \text{slope of} \\ f \text{ at } a}}{f'(a)} = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \underbrace{\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}}_{\text{slope of } f \text{ at } a}$.

Consider $\lim\limits_{x\to a}[f(x)-f(a)]$ on its own

Then $\lim\limits_{x\to a}[f(x)-f(a)]\left(\dfrac{x-a}{x-a}\right) \leftarrow$ multiply by a clever 1

$$= \underbrace{\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)}(x-a)$$

Note, this limit exists and is $f'(a)$ by our initial assumption, i.e., it is a number.

Also, $\lim\limits_{x\to a}(x-a) = a-a = 0 \leftarrow$ 0 is a number

Since $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists and $\lim\limits_{x\to a}(x-a)$ exists, we have by a limit property

that $\lim\limits_{x\to a}\left(\dfrac{f(x)-f(a)}{x-a}\right)(x-a) = \left(\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\right)\left(\lim\limits_{x\to a}(x-a)\right)$

$$= (f'(a))\cdot(0) = 0$$

So, $\lim\limits_{x\to a}[f(x)-f(a)] = 0$

$\Rightarrow \quad 0 = \lim\limits_{x\to a}[f(x)-f(a)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} f(a)$

$$= \lim_{x\to a} f(x) - f(a) = 0$$

$$\Rightarrow \lim_{x\to a} f(x) = f(a).$$

So f is continuous at $x=a$.

□

"QED"

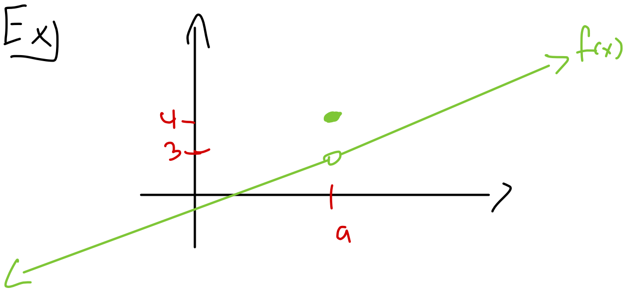
$\Rightarrow$ Latin abbreviation for "it has been shown."

We proved differentiable implies continuity.

The contrapositive of any true statement is true

$\Rightarrow$ If $f(x)$ is not continuous at $x=a$, then it is not differentiable at $x=a$.

Ex)



Note, $f(x)$ is not continuous at $x=a$, as $\lim_{x\to a} f(x) = 3 \neq 4 = f(a)$.

Thus, $f(x)$ is not differentiable at $x=a$.

Note, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ exists everywhere except at $x=a$.

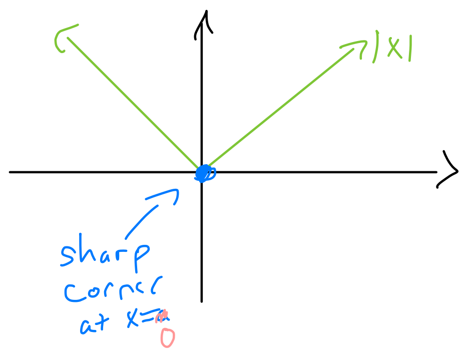
So, we say f is differentiable on $(-\infty, a) \cup (a, \infty)$

or $x \neq a$

Question. *True or False: If f is continuous, then f is differentiable.*

(this is the converse of our theorem, and converses may or may not be true)

False!



$f(x) = |x|$ is continuous at $x=a$, but f is not differentiable at $x=a$.

2.8.2 Higher Order Derivatives

Since f' is a function, there is nothing stopping us from taking the derivative of f'.
Notation:

The second derivative of f is notated by

$$f''(x) = \frac{d^2 f}{dx^2}$$

why $\frac{d^{\boxed{2}}f}{dx^{\boxed{2}}}$? Note $\frac{d}{dx}$ is an operation telling us to take a derivative with respect to x.

$$\Rightarrow f'(x) = \frac{d}{dx} f = \frac{df}{dx}$$

$$\Rightarrow f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\frac{df}{dx} = \frac{d^2 f}{(dx)^2}$$

$$\underbrace{f'''(x) = \frac{d^3 f}{dx^3}}_{\text{third derivative}}, \quad \underbrace{f^{(4)}(x) = \frac{d^4 f}{dx^4}}_{\text{fourth derivative}}, \ldots$$

Example. Find $f''(x)$ if ~~$f(x) = x^3 - x$.~~ $\Rightarrow f''(x) = 6x \cancel{-1}$

$f(x) = x^2 - x$, find $f''(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2 - (x+h)] - [x^2 - x]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 \cancel{-x} - h \cancel{-x^2} \cancel{+x}}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 - h}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}}$$

$$= 2x + 0 - 1 = 2x - 1$$

So $f'(x) = 2x - 1$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{[2(x+h) - 1] - [2x - 1]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2h \cancel{-1} - \cancel{2x} \cancel{+1}}{h} = \lim_{h\to 0} \frac{2h}{h}$$

$$= \lim_{h\to 0} 2$$

$$= 2$$

So $f''(x) = 2$.

$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{2-2}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Example Continued: